# Strictly ascending HNN extensions in soluble groups

J. O. Button

Selwyn College

University of Cambridge

Cambridge CB3 9DQ

U.K.

jb128@dpmms.cam.ac.uk

**Abstract**

We show that there exist finitely generated soluble groups which are not LERF but which do not contain strictly ascending HNN extensions of a cyclic group. This solves Problem 16.2 in the Kourovka notebook. We further show that there is a finitely presented soluble group which is not LERF but which does not contain a strictly ascending HNN extension of a polycyclic group.

## 1   Introduction

A group is said to be extended residually finite or **ERF** if every subgroup is an intersection of finite index subgroups. This is a vast strengthening of the property of residual finiteness. Any virtually polycyclic group is ERF, as first proved by Mal'cev. Other groups can be ERF; for instance see [14] for recent results, but not a single other example of a finitely generated ERF group is known. This is Question 8 in [7], although it dates back to the paper [9] where it is shown that a finitely generated virtually soluble group which is ERF must be virtually polycyclic. This was reproved in [1] and [13]. Indeed the latter paper extends this "ERF implies virtually polycyclic" result to a wider class of finitely generated groups which includes

all finitely generated linear and all finitely generated elementary amenable groups. To further illustrate the problem, the class of ERF groups is closed under taking quotients and subgroups, but no member can contain a non-abelian free group. Thus a new finitely generated ERF group would not be elementary amenable but nor could it contain a non-abelian free group, and even residually finite examples of this are hard to come by.

Another property that is considerably stronger than residual finiteness but weaker than ERF is that of being locally extended residual finite or **LERF**. This is when every finitely generated subgroup is the intersection of finite index subgroups and is sometimes called subgroup separable (although the phrase subgroup separable originally referred to ERF, as can be seen in older papers). This property is better suited to finitely generated groups and is useful not just in group theory but in 3-manifold topology, because it has important consequences when the group is the fundamental group of a compact 3-manifold and the finitely generated subgroup is that of a surface. Not all compact 3-manifolds have LERF fundamental group but it is an open question as to whether hyperbolic 3-manifolds do. However a classical result of M. Hall Jnr. is that finitely generated free groups are LERF and this was extended to closed surface groups by P. Scott in [15]. It is clear that the LERF property is closed under taking subgroups as well as finite index supergroups (for the latter claim see [14] Lemma 4.2, which establishes this for ERF and the proof generalises immediately for LERF). However a quotient of a LERF group need not be LERF (this follows immediately for any property which holds for finitely generated free groups but not for all finitely generated groups).

As for determining that a group $G$ is not LERF, a useful method is due to Blass and Neumann in [4]. This says that if $G$ contains a subgroup $H = \langle t, B \rangle$ which is a strictly ascending HNN extension of a finitely generated group $B$ with stable letter $t$ then $G$ is not LERF because in a finite quotient of $G$ we must have $B$ and $tBt^{-1} < B$ going to conjugate subgroups, thus they are of the same order and hence equal, so we cannot separate $B$ from $tBt^{-1}$. The strictly ascending condition means that if $A_1$ and $A_2$ are the associated subgroups of the HNN extension then one is equal to the base $B$ and the other is strictly contained in $B$, so that conjugation by $t$ induces an injective but not surjective endomorphism of $B$.

However we can ask if this always detects the absence of LERF in a finitely generated group. In particular we can examine soluble groups $S$ and the subgroups of $S$ which are HNN extensions. The advantage of solubility is

that any HNN extension contained in $S$ is ascending, meaning that at least one of $A_1$ and $A_2$ is equal to $B$ (as otherwise $S$ would contain a non-abelian free group). If $A_1 = A_2 = B$ then we have a semidirect product $B \rtimes \mathbb{Z}$, otherwise if $B$ is finitely generated we can conclude that $S$ is not LERF.

In [10] it is asked if a finitely generated soluble group which is not LERF must contain a strictly ascending HNN extension of a cyclic group. We show that this is not the case and we show further that there exists a finitely presented soluble group which is not LERF but which does not contain a strictly ascending HNN extension of a finitely generated abelian group, or even a polycyclic group.

# 2   Strictly ascending HNN extensions

We summarise the facts we will need about ascending and strictly ascending HNN extensions. If $B$ is a group with an isomorphism $\theta$ to a subgroup $A$ of $B$ then the **ascending HNN extension** $G = \langle t, B \rangle$ is formed by adjoining to $G * \langle t \rangle$ the relations $tbt^{-1} = \theta(b)$ over all $b \in B$ (or just over a generating set for $B$). This gives rise to the associated homomorphism $\chi$ of the HNN extension which is defined by $\chi(t) = 1$ and $\chi(B) = 0$. If $A = B$ then $\ker(\chi)$ is equal to $B$ and $G$ is the semidirect product $B \rtimes_\theta \mathbb{Z}$, but if $tBt^{-1} = A < B$ then $\ker(\chi)$ is a strictly ascending union $\cup_{i=0}^{\infty} t^{-i} B t^i$ of subgroups which are all isomorphic to $B$. This means that the kernel must be infinitely generated; indeed if $C$ is any finitely generated subgroup of ker $(\chi)$ then it must be contained in $t^{-i} B t^i$ for some $i$, and so $C$ is conjugate to a finitely generated subgroup of $B$.

Any $g \in G$ can be expressed in the form $g = t^{-k} b t^l$ for some $b \in B$ and $k, l \geq 0$, thus $\chi(g) = l - k$. Moreover if $B$ is soluble then so is $\ker(\chi)$ because solubility is a local condition, and also $G$ which is equal to $\ker(\chi) \rtimes_\alpha \mathbb{Z}$ where $\alpha$ is the automorphism of $\ker(\chi)$ that is induced under conjugation by $t$. A strictly ascending HNN extension of a cyclic group must imply that the cyclic group is the integers $\mathbb{Z}$ and it will be isomorphic to the soluble Baumslag-Solitar group $BS(1, m) = \langle x, y | yxy^{-1} = x^m \rangle$ for $m \neq 0, \pm 1$.

Now suppose that $BS(1, m) \leq G = \langle t, B \rangle$ where $G$ is itself a strictly ascending HNN extension. On trying to locate the elements $x, y$ in $G$ we see by applying $\chi$ that $x \in \ker(\chi)$ and so, on conjugating $\langle x, y \rangle$ by an appropriate power of $t$, we have $x \in B$ and $y = t^{-k} b t^l$ for $b \in B$ and $k, l \geq 0$. But we can further conjugate $BS(1, m)$ by $t^k$ to get an element $x = a$ of infinite order and

$y = bt^n$ for $a, b \in B$ and $n = l - k$. Thus from the Baumslag-Solitar relation we must have $\theta^n(a) = b^{-1}a^m b$ if $n \geq 0$ and $\theta^{-n}(b^{-1}a^m b) = a$ otherwise.

**Theorem 2.1** *The soluble group $G = \langle t, \mathbb{Z} \times \mathbb{Z} \rangle$, which is an ascending HNN extension formed by taking the endomorphism $\theta(u) = u^5 v^{-1}, \theta(v) = u^2$ where $u, v$ is the standard generating pair for $\mathbb{Z} \times \mathbb{Z}$, is not LERF but does not contain a strictly ascending HNN extension of a cyclic group.*

**Proof.** The endomorphism $\theta$ of $\mathbb{Z} \times \mathbb{Z}$ gives rise to a matrix with determinant 2, so $\theta$ is injective but not surjective. Thus $G$ is soluble, but not LERF by the Blass-Neumann result. Now suppose $G$ contains $BS(1, m)$ for $m \neq 0, \pm 1$. By the comment before the theorem, we have here that the base $B = \mathbb{Z} \times \mathbb{Z}$ and so we obtain $a \in B - \{0\}$ and $m \in \mathbb{Z} - \{0, \pm 1\}$ such that $\theta^n(a) = a^m$ for $n > 0$ (or $\theta^{-n}(a^m) = a$ for $n < 0$). Writing this additively and regarding $\theta$ as an invertible linear map of $\mathbb{R}^2$, we are claiming (in both cases) that $\theta^n$ has the eigenvalue $m$, so $\theta$ has an eigenvalue $\lambda \in \mathbb{C}$ where $\lambda^n = m$.

Now the product of the two eigenvalues $\lambda, \mu$ of $\theta$ is 2 and the sum is 5, so $\lambda$ and $\mu = 2/\lambda$ satisfy $x^2 - 5x + 2 = 0$ and are algebraic integers. This means that $\mu^n$ is too for $n > 0$ and if $n < 0$ then we are done because $\lambda$ being an algebraic integer would imply that $\lambda^{-n} = 1/m$ would be too. But we also have $m\mu^n = 2^n$. This means that $\mu^n$ is in $\mathbb{Q}$ thus is an integer dividing $2^n$. Now we have $1 \leq |\mu| \leq 2$ and $|\lambda + \mu| = |2/\mu + \mu| \leq 4$ but this contradicts $\lambda + \mu = 5$.

$\square$

Problem 16.2 in [10] asks whether a finitely generated solvable group is LERF if and only if it does not contain $BS(1, m)$ for $m > 1$. Thus we see from Theorem 2.1 that the answer is no. However this group contains (and indeed is) a strictly ascending HNN extension of a finitely generated abelian group, and we might generalise the question by asking for a finitely generated non-LERF soluble group which does not contain subgroups of this sort. To make further progress we have the following proposition which is the main tool we will be using.

**Proposition 2.2** *Suppose that $G$ is a group possessing a homomorphism onto an abelian group $A$ with kernel $K$ and $H = \langle t, B \rangle$ is a subgroup of $G$ which is a strictly ascending HNN extension with base $B$ and stable letter $t$. Then the subgroup $S = \langle t, B \cap K \rangle$ is also a strictly ascending HNN extension.*

**Proof.** We have that $tBt^{-1} < B$. As $K$ is normal in $G$, we certainly have $tCt^{-1} \leq C$ where $C = B \cap K$. Suppose this is actually equality. We take any element $b \in B$ and consider $b_0 = tbt^{-1}$. This is also in $B$ and $\theta(b_0) = \theta(b)$ as $A$ is abelian. Thus $bb_0^{-1} \in C$ so by assumption we have $c_0 \in C$ with $tc_0t^{-1} = bb_0^{-1}$. Then $tc_0bt^{-1} = b$ and $c_0b \in B$, giving the contraction $tBt^{-1} = B$.

$\square$

**Theorem 2.3** *There exists a finitely presented soluble group which is not LERF but which does not contain a strictly ascending HNN extension of a finitely generated abelian group.*

**Proof.** The particular group $G$ we work with is the Baumslag-Remeslennikov group. This is a finitely presented soluble group which is residually finite; indeed it is linear over $\mathbb{R}$. It is formed in the following way: let $W$ be the wreath product $\mathbb{Z} \wr \mathbb{Z}$ which can be thought of as the semidirect product $\mathbb{Z}^\infty \rtimes_\alpha \mathbb{Z}$, where $\mathbb{Z}^\infty$ is the free abelian group on countably many generators $a_i$ for $i \in \mathbb{Z}$ and $\alpha$ acts as a shift up by 1 so that the stable letter $s$ generating $\mathbb{Z}$ satisfies $sa_is^{-1} = a_{i+1} = \alpha(a_i)$. Now $W$ is finitely generated but not finitely presented. However we consider the endomorphism $\theta$ of $W$ defined by $\theta(s) = s$ and $\theta(a_i) = a_ia_{i+1}$. This is injective as any element of $W$ can be expressed uniquely as $as^i$ for $a \in \mathbb{Z}^\infty$ and $i \in \mathbb{Z}$. However it is not surjective: if we let the degree $d(a)$ of a non-zero element $a \in \mathbb{Z}^\infty$ be the difference in the indices of the highest non-zero entry of $a$ and the lowest then $d(\theta(a)) = d(a) + 1$, and so $a_i$ has no preimage under $\theta$ as $d(a_i) = 1$.

We then use $\theta$ to form $G = \langle t, W \rangle$ which is the strictly ascending HNN extension with base $W$ and stable letter $t$, so that $twt^{-1} = \theta(w)$ for $w \in W$. Consequently any element of $G$ can be written as $t^{-m}as^it^n$ for $a \in \mathbb{Z}^\infty$, $i \in \mathbb{Z}$ which is uniquely defined and $m, n \geq 0$. Clearly $G$ is not LERF because $W$ is finitely generated. Let us suppose that $G$ contains $H = \langle \tau, B \rangle$ which is a strictly ascending HNN extension with finitely generated abelian base $B$ and stable letter $\tau$. Now we have the homomorphism from $G$ to $\mathbb{Z}$ associated to the decomposition of $G$ as $\langle t, W \rangle$ with kernel $K$. By Proposition 2.2 we can replace $B$ with $B \cap K$ (which we will henceforth call $B$) because $B \cap K$ must also be finitely generated abelian, and $\langle \tau, B \cap K \rangle$ becomes the new $H$. But $K$ is an ascending union and $B$ is finitely generated so is contained in $t^{-i}Wt^i$ for some $i \geq 0$. We can therefore replace $H$ with $t^iHt^{-i} = \langle t^i\tau t^{-i}, t^iBt^{-i} \rangle$ which

is still a strictly ascending HNN extension where the base $t^i B t^{-i}$ (which we now also rename $B$) lies in $W$.

However there is also a homomorphism from $G$ to $\mathbb{Z}$ given by the exponent sum of $s$ in an element $g$ of $G$, and another application of Proposition 2.2 means we now think of the base $B$ of $H$ as lying in $W$ with exponent sum of $s$ equal to 0, which means $B$ lies in $\mathbb{Z}^\infty$. Thus we must have some $g \in G$ with $gBg^{-1} < B < \mathbb{Z}^\infty$. We let $g = t^{-m}as^i t^n$ as above. Remembering that $t$ commutes with $s$, $\mathbb{Z}^\infty$ is abelian and $t\mathbb{Z}^\infty t^{-1} < \mathbb{Z}^\infty$, we obtain

$$s^i B s^{-i} < t^{m-n} B t^{-(m-n)} \text{ and } t^{n-m} B t^{-(n-m)} < s^{-i} B s^i.$$

We cannot have $n = m$ as $B$ is not strictly contained in $s^{-i} B s^i$ for any $i$. If $n > m$ then we use the second formula and take an element $b$ in $B$ which has maximum degree. This exists because $B$ is finitely generated. Conjugating $b$ by positive powers of $t$ will increase the degree but shifting $b$ will keep it constant. Therefore $t^{n-m} b t^{-(n-m)}$ cannot lie in $s^{-i} B s^i$. On the other hand, if $n < m$ then we use the first formula and take an element $b$ of $B$ with minimum degree. Again one must exist unless $B = \{0\}$, in which case we do not have strict containment. Because the degree of all non-zero elements of $B$ must increase under conjugation by $t^{m-n}$, we see that $s^i b s^{-i}$ cannot be contained in the right hand side.

$\square$

Note that the use of Proposition 2.2 in the proof of Theorem 2.3 means that the only point where we used the fact that the finitely generated base $B$ was abelian was to conclude that $B \cap K$ was also finitely generated. Thus Theorem 2.3 applies without change of proof to show that $G$ does not contain a strictly ascending HNN extension where the base is any finitely generated group all of whose subgroups are finitely generated. Now the soluble groups with this property are precisely the polycyclic groups, so we have the following Corollary.

**Corollary 2.4** *The Baumslag-Remeslennikov finitely presented soluble group is not LERF but does not contain a strictly ascending HNN extension of a polycyclic group.*

# 3   Further comments

We have seen that there exist finitely generated (and even finitely presented) soluble groups $G$ where the failure of $G$ to be LERF cannot be witnessed by using the Blass-Neumann result to find a strictly ascending HNN extension with a finitely generated base that is cyclic, abelian or even polycyclic. But the possibility still remains that this result will always show the absence of LERF because of the existence of some strictly ascending HNN extension of an arbitrary finitely generated subgroup (which of course will necessarily be soluble). Thus the following question remains.

**Question 3.1** *If $G$ is a finitely generated (or finitely presented) soluble group which is not LERF then must it contain a strictly ascending HNN extension of some finitely generated group?*

One may want to include $G$ being residually finite in the hypothesis in case of counterexamples which might be so nasty as to be far from being residually finite (and hence even further from being LERF) as well as being unable to contain strictly ascending HNN extensions. It is remarked in [8] after Proposition 3.19 that it would be interesting to characterise LERF groups amongst finitely generated soluble groups but this is open even when restricted to metabelian groups. The Proposition itself shows that the (standard) wreath product $A \wr B$ is LERF when $A$ is finitely generated abelian and $B = \mathbb{Z}$. This was extended in [2] to when both $A$ and $B$ are finitely generated abelian.

We remark though that the answer to the above question when extended to arbitrary finitely presented groups is a definite no, even in a class of finitely presented groups which is regarded as generally well behaved. This is the class of fundamental groups of 3-manifolds, and if the 3-manifold is compact then the fundamental group is finitely presented. Theorem 4.1 of [6] states that the fundamental group of any (not necessarily compact) 3-manifold cannot contain a strictly ascending HNN extension with finitely generated base. However there are certainly compact 3-manifolds $M$ whose fundamental group is not LERF: the first was given in [5], giving a residually finite and coherent counterexample $\pi_1(M)$.

We can give one class of soluble groups where the answer to Question 3.1 is yes: these are the constructible soluble groups first introduced in [3]. A group is constructible if it has a subgroup of finite index which can be built up as an HNN extension (or amalgamated free product) where the base and associated subgroups (or the factors and the amalgamated subgroup)

have previously been built in this way. All constructible groups are finitely presented, although not vice versa (for instance the Baumslag-Remeslennikov example in Section 2). If a constructible group is soluble then, because of the ubiquity of free groups in HNN extensions and amalgamated free products, we can drop the amalgamated free product construction without loss. As for HNN extensions, we may assume that they are all ascending. Consequently constructible groups that are soluble are especially well behaved: they are all residually finite and even linear over $\mathbb{Q}$, virtually torsion free and of finite Prüfer rank, for instance see [11] Subsection 11.2. Moreover it is clear why the answer to Question 3.1 is straightforward here. Either we never use a strictly ascending HNN extension in building our group, but then we are always taking finite extensions or the semidirect product by $\mathbb{Z}$ with a group previously obtained, thus staying in the class of virtually polycyclic groups (and polycyclic groups if the result is soluble) which are LERF. Otherwise we will use a strictly ascending HNN extension for the first time which means that here the base is polycyclic, and this will be contained in our final group.

We remark that the finitely generated subgroups of (virtually) soluble constructible groups are precisely the finitely generated, residually finite (virtually) soluble groups of finite Prüfer rank. This is Theorem A of [3] without addition of the word virtually. Otherwise suppose that $G$ is a finitely generated residually finite group of finite Prüfer rank with a normal index $m$ subgroup $N$ which is soluble, and $C$ is a soluble constructible group containing $N$. Then, as $N$ and $C$ are contained in $GL(n, \mathbb{Q})$ for some $n$, we have (by [16] Lemma 2.3 say) that $G$ is a subgroup of $GL(mn, \mathbb{Q})$ with $G/N$ isomorphic to a group $P$ of $m \times m$ permutation matrices. Then $G$ embeds in the subgroup $S$ of $GL(mn, \mathbb{Q})$ which is the extension by $P$ of the group made up of $m$ diagonal blocks, each with an entry in $C$. But $S$ is constructible as it has an index $m$ subgroup isomorphic to $C \times \ldots \times C$ ($m$ times) and the direct product of constructible groups is constructible by [3] Proposition 2(b).

These groups occur in various places: for instance the finitely generated residually finite groups with polynomial subgroup growth are precisely those which are virtually soluble of finite Prüfer rank, see [12]. However even if we have a group where the absence of LERF can be detected because it contains a strictly ascending HNN extension of a finitely generated group, this need not be true of its subgroups. For instance we can take $\pi_1(M)$ above and form the free (or direct) product $P$ of $\pi_1(M)$ with a strictly ascending HNN extension of a finitely generated group, whereupon both $P$ and $\pi_1(M)$ fail to be LERF but only $P$ contains such an HNN extension.

# References

[1] R. C. Alperin: Solvable groups of exponential growth and HNN-extensions. In: Groups – Korea '96, pp. 1–5. Pusan (de Gruyter, Berlin) (2000)

[2] R. C. Alperin: Metabelian Wreath Products are LERF. http://arxiv.org/abs/math/0609611 (2006)

[3] G. Baumslag and R. Bieri: Constructable Solvable Groups. Math. Z. **151**, 249-257 (1976)

[4] A. Blass and P. M. Neumann: An application of universal algebra in group theory. Michigan Math. J. **21**, 167–169 (1974)

[5] R. G. Burns, A. Karrass and D. Solitar: A note on groups with separable finitely generated subgroups. Bull. Austral. Math. Soc. **36**, 153–160 (1987)

[6] J. O. Button: Mapping tori with first Betti number at least two. J. Math. Soc. Japan **59**, 351–370 (2007)

[7] J. O. Button: Largeness of LERF and 1-relator groups. Groups Geom. Dyn., to appear (2010)

[8] Y. de Cornelier: Finitely presented wreath products and double coset decompositions. Geom. Dedicata **122**, 89–108 (2006)

[9] S. C. Jeanes and J. S. Wilson: On finitely generated groups with many profinite-closed subgroups. Arch. Math. (Basel) **31**, 120–122 (1978/79)

[10] The Kourovka notebook: Unsolved problems in group theory. Sixteenth edition, Edited by V. D. Mazurov and E. I. Khukhro. Russian Academy of Sciences Siberian Division, Institute of Mathematics, Novosibirsk (2006)

[11] J. C. Lennox and D. J. S. Robinson: The theory of infinite soluble groups. Oxford Mathematical Monographs. Oxford University Press, Oxford (2004)

[12] A. Lubotzky and D. Segal: Subgroup growth. Progress in Mathematics 212, Birkhaüser Verlag, Basel (2003)

[13] D. J. S. Robinson, A. Russo and G. Vincenzi: On groups which contain no HNN-extensions. Internat. J. Algebra Comput. **17**, 1-11 (2007)

[14] D. J. S. Robinson, A. Russo and G. Vincenzi: On groups whose subgroups are closed in the profinite topology. J. Pure Appl. Algebra **213**, 421–429 (2009)

[15] P. Scott: Correction to: "Subgroups of surface groups are almost geometric". J. London Math. Soc. **32**, 217–220 (1985)

[16] B. A. F. Wehrfritz: Infinite linear groups. Ergebnisse der Matematik und ihrer Grenzgebiete, Band 76. Springer-Verlag, New York-Heidelberg (1973)